

MAKING OLD SEMINAL RESULTS WORLD-WIDE AVAILABLE !**FORWARD**

The seminal paper of Crum, published in 1955, is now a standard reference in nonlinear science and supersymmetric quantum mechanics. It introduces the Crum transformations, a cornerstone of integrability and a beautiful generalization of Darboux transformations.

Since I am sure that many people would like to study carefully this masterpiece I offer here a LaTeX version of the paper. The purpose is to prevent all sorts of rediscoveries and promote real progress. I did very minor changes with respect to the old published version. The most important was to put the list of references at the end and not as footnotes. Crum's paper has 7 points. The first point is the statement of Crum's theorem, i.e., the possibility to write the solutions of a tower of so-called associated Sturm-Liouville (SL) systems (all of them Dirichlet from the point of view of boundary conditions) as a quotient of Wronskian determinants. The second point refers to the first associated SL system, dealing in fact with the SL Darboux transformations. Points 3 and 4 are a detailed study of the higher order associated SL systems (SL supersymmetric partners). Point 5 contains four noted applications. The corollary of Crum's theorem is at point 6. Finally, point 7 states the possibility to build a regular SL system with any finite set of real numbers as eigenvalues, starting from a given associated SL system, a remarkable general result.

H C R

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ASSOCIATED STURM-LIOUVILLE SYSTEMS

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1. Let the regular Sturm-Liouville system

$$\begin{cases} y'' + [\lambda - q(x)]y = 0 & (0 < x < 1) , \\ y'(0) = h^{(0)}y(0) , & y'(1) = h^{(1)}y(1) \end{cases} \quad \begin{matrix} (A) \\ (B) \end{matrix}$$

have eigenvalues $\lambda_0 < \lambda_1 < \lambda_2$, etc, and eigenfunctions ϕ_s corresponding to λ_s . Let $q(x)$ be repeatedly differentiable in $(0,1)$; then the ϕ_s also are repeatedly differentiable; let W_{ns} be the Wronskian of the $n+1$ functions $\phi_0, \phi_1, \dots, \phi_{n-1}, \phi_s$ and let W_n be the Wronskian of the n functions $\phi_0, \phi_1, \dots, \phi_{n-1}$. Then, if $n \geq 1$ and

$$\phi_{ns} = W_{ns}/W_n ,$$

the functions ϕ_{ns} ($s \geq n$) are the eigenfunctions, with eigenvalues λ_s , of the system

$$\begin{cases} y'' + [\lambda - q_n(x)]y = 0 & (0 < x < 1) , \\ \lim_{x \rightarrow 0} y(x) = 0 , & \lim_{x \rightarrow 1} y(x) = 0 \end{cases} \quad \begin{matrix} (A_n) \\ (B_n) \end{matrix}$$

where

$$q_n(x) = q(x) - 2 \frac{d^2}{dx^2} \log W_n . \quad (C_n)$$

For $n = 1$, the system (A_n, B_n) is regular; but, for $n > 1$,

$$q_n(x) \approx \begin{cases} n(n-1)x^{-2} & (x \rightarrow 0) , \\ n(n-1)(1-x)^{-2} & (x \rightarrow 1) \end{cases}$$

Inside $(0,1)$, W_n is non-zero and q_n is continuous. For $s < n$, $\phi_{ns} \equiv 0$; for $s > n$, ϕ_{ns} has exactly $s - n$ zeros inside $(0,1)$. The family ϕ_{ns} ($s \geq n$) is L^2 -closed and complete over $(0,1)$.

The system (A_n, B_n) may be called the 'nth system associated with the system (A, B) '. In this note the above statements are established, and examples are given of systems associated with non-regular Sturm-Liouville systems.

If $q(x)$ is continuous but not differentiable, the ϕ_s are differentiable twice only, and the Wronskians do not exist; however, when the Wronskians W_{ns} , W_n exist, they are equal to the modified Wronskians W_{ns}^* , W_n^* obtained by replacing $\phi_s^{(2k)}$ by $(-\lambda_s)^k \phi_s$, and $\phi_s^{(2k+1)}$ by $(-\lambda_s)^k \phi_s'$; the W_n^* are at least twice differentiable, and the statements above are true for non-differentiable continuous q provided that the W are replaced by W^* .

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2. The case $n = 1$

We have $W_1 = \phi_0$, of constant sign [1] for $0 \leq x \leq 1$; and

$$\phi_{1s} = \phi'_s - \frac{\phi'_0}{\phi_0} \phi_s = \phi'_s - v \phi_s, \text{ say,} \quad (D_1)$$

where

$$v' + v^2 = q - \lambda_0. \quad (E)$$

Then

$$\frac{d}{dx} (\phi_0 \phi_{1s}) = \phi_0 \phi''_s - \phi''_0 \phi_s = (\lambda_0 - \lambda_s) \phi_0 \phi_s. \quad (F_1)$$

Since

$$\phi_{1s}(0) = 0 = \phi_{1s}(1), \quad (G)$$

we have

$$\phi_0 \phi_{1s} = (\lambda_0 - \lambda_s) \int_0^x \phi_0(\xi) \phi_s(\xi) d\xi = -(\lambda_0 - \lambda_s) \int_x^1 \phi_0 \phi_s d\xi. \quad (G')$$

Hence

$$\begin{aligned} \phi'_{1s} &= (\lambda_0 - \lambda_s) \phi_s - v \phi_{1s}, \\ \phi''_{1s} &= (\lambda_0 - \lambda_s) \phi'_s - v' \phi_{1s} - v[(\lambda_0 - \lambda_s) \phi_s - v \phi_{1s}] \\ &= (\lambda_0 - \lambda_s - v' + v^2) \phi_{1s} \\ &= (q_1 - \lambda_s) \phi_{1s}, \end{aligned}$$

where

$$q_1 = \lambda_0 - v' + v^2 = q - 2v' = q - 2 \frac{d^2}{dx^2} (\log W_1).$$

Now from (D_1) ,

$$\phi_{1s}/\phi_0 = \frac{d}{dx} (\phi_s/\phi_0);$$

since ϕ_s has exactly s zeros [1] inside $(0,1)$, by Rolle's theorem, ϕ_{1s} has at least $s - 1$. But from (F_1) and (G) and Rolle's theorem, ϕ_{1s} has at most $s - 1$ zeros inside $(0,1)$; hence it has $s - 1$ exactly. it follows [1] that the ϕ_{1s} ($s \geq 1$) are all the eigenfunctions of the regular system (A_1, B_1) . For $\lambda \neq \lambda_0$ the general solution of (A_1) is

$$X_1 = W(\phi_0, \chi)/W_1,$$

where χ is the general solution of (A) . For $\lambda = \lambda_0$, $W(\phi_0, \chi)$ is constant and one solution of (A_1) is $1/\phi_0$; two independent solutions are

$$\frac{1}{\phi_0} \int_0^x \phi_0^2(\xi) d\xi, \quad \frac{1}{\phi_0} \int_x^1 \phi_0^2(\xi) d\xi.$$

It is easily verified that the only solutions of (A_1) which satisfies (G) are the ϕ_{1s} ($s \geq 1$).

3. The case $n > 1$

Applying Jacobi's theorem to the determinant W_{ns} , we have, for $n > 1$,

$$W_{ns}W_{n-1} = W_n \frac{d}{dx} W_{n-1,s} - W_{n-1,s} \frac{d}{dx} W_n ,$$

with a similar relation with W^* for W . Hence

$$\begin{aligned} \phi_{ns} &= \frac{W_{ns}}{W_n} = \frac{1}{W_{n-1}} \frac{d}{dx} (W_{n-1} \phi_{n-1,s}) - \phi_{n-1,s} \frac{1}{W_n} \frac{d}{dx} W_n \\ &= \phi'_{n-1,s} - v_{n-1} \phi_{n-1,s} = \frac{1}{\phi_{n-1,n-1}} W(\phi_{n-1,n-1}, \phi_{n-1,s}) , \end{aligned} \quad (D_n)$$

where

$$v_n = \phi'_{nn}/\phi_{nn} , \quad v_{n-1} = W'_n/W_n - W'_{n-1}/W_{n-1} .$$

Hence, by steps similar to those of §2, and by induction on n ,

$$v'_n + v_n^2 = q_n - \lambda_n , \quad (E_n)$$

$$\frac{d}{dx} (\phi_{n-1,n-1} \phi_{ns}) = (\lambda_{n-1} - \lambda_s) \phi_{n-1,n-1} \phi_{n-1,s} , \quad (F_n)$$

$$\phi''_{ns} = (q_n - \lambda_s) \phi_{ns} , \quad q_n = q_{n-1} - 2v'_{n-1} ,$$

$$q_n + 2 \frac{d}{dx} \left(\frac{W'_n}{W_n} \right) = q_{n-1} + 2 \frac{d}{dx} \left(\frac{W'_{n-1}}{W_{n-1}} \right) = q .$$

We now prove by induction on n the following:

$$\phi_{ns} = C_{ns} \prod_{t=0}^{n-1} (\lambda_t - \lambda_s) x^n [1 + O(x^2)] \quad (C_{ns} \neq 0) , \quad (G_n)$$

$$\phi'_{ns} = nx^{-1} \phi_{ns} [1 + O(x^2)] , \quad (H_n)$$

$$v_n = nx^{-1} [1 + O(x^2)] , \quad (J_n)$$

all as $x \rightarrow 0$, with similar relations as $x \rightarrow 1$;

$$\phi_{ns} \text{ has } s - n \text{ zeros inside } (0,1) . \quad (K_n)$$

By (K_n) , ϕ_{nn} , and so also W_{n+1} , is non-zero inside $(0,1)$, so that q_{n+1} and $\phi_{n+1,s}$ are continuous inside $(0,1)$. First, by (G) and (G') , as $x \rightarrow 0$,

$$\phi_{1s}(x) \sim (\lambda_0 - \lambda_s) \phi_s(0) x ;$$

also

$$\phi_{1s}''(0) = (q_1 - \lambda_s)\phi_{1s}(0) = 0 ,$$

which together imply (G_1) ; (H_1) follows from (G_1) and (F_1) , together with

$$\phi_s = \phi_s(0)[1 + h^{(0)}x + O(x^2)] ;$$

and (J_1) is a case of (H_1) . It remains to deduce (G_{n+1}) to (K_{n+1}) from (G_n) to (K_n) . First, by (D_{n+1}) , (H_n) , (J_n) ,

$$\phi_{n+1,s} = \phi_{ns} \left[\frac{n}{x} + O(x) - \frac{n}{x} + O(x) \right] = o(1) \quad (x \rightarrow 0) .$$

Hence

$$\phi_{nn}\phi_{n+1,s} = (\lambda_n - \lambda_s) \int_0^x \phi_{nn}\phi_{ns}d\xi ,$$

whence we have (G_{n+1}) with

$$C_{n+1,s} = C_{ns}/(2n+1) \neq 0 .$$

By differentiating this last we obtain (H_{n+1}) , of which (J_{n+1}) is a special case.

From (D_{n+1}) and (K_n) , $\phi_{n+1,s}$ has at least $s - n - 1$ zeros inside $(0,1)$; from (F_{n+1}) , (K_n) , (G_n) , it has at most $s - n - 1$ zeros inside $(0,1)$; hence (K_{n+1}) is deduced.

Lastly we may prove that, as $x \rightarrow 0$,

$$q_n(x) = n(n-1)x^{-2} + O(1) , \tag{L_n}$$

with a similar relation as $x \rightarrow 1$. For, given (L_n) and (J_n) ,

$$q_{n+1} = q_n - 2v_n' = 2\lambda_n + 2v_n^2 - q_n = O(1) + n(n+1)x^{-2} ,$$

which is (L_{n+1}) .

For $\lambda \neq \lambda_s$ ($s < n$) the general solution of (A_n) is

$$y = \chi_n = W(\phi_0, \phi_1, \dots, \phi_{n-1}, \chi)/W_n ,$$

where χ is the general solution of (A) . For $\lambda = \lambda_{n-1}$ a solution is

$$y = \frac{1}{\phi_{n-1,n-1}} W(\phi_{n-1,n-1}, \chi_{n-1,n-1}) = \frac{C}{\phi_{n-1,n-1}} = C \frac{W(\phi_0, \phi_1, \dots, \phi_{n-2})}{W(\phi_0, \phi_1, \dots, \phi_{n-1})} .$$

For $\lambda = \lambda_s$, $s \leq n-1$, a solution is

$$y = \psi_{ns} = W_n^{(s)}/W_n ,$$

where $W_n^{(s)}$ is the Wronskian of the $n-1$ functions

$$\phi_t \quad (0 \leq t \leq n-1 ; t \neq s) .$$

4. Since the system (A_n, B_n) is not regular for $n > 1$, it remains to prove that the family ϕ_{ns} ($s \geq n$) is L^2 -complete over $(0,1)$; this implies incidentally that the ϕ_{ns} are the only bounded solutions of (A_n) . Since (A_1, B_1) is regular, it is sufficient to verify that the completeness of the family ϕ_{ns} implies that of the family $\phi_{n+1,s}$.

Let $f(x)$ be of $L^2(0,1)$; then, given $\epsilon > 0$, there exists $g(x)$ such that

(i) $g(x) = 0$ ($0 < x < \delta$; $1 - \delta < x < 1$; $\delta > 0$),

(ii) $g'(x)$ is continuous in $(0,1)$,

(iii) $\int_0^1 |f - g|^2 d\xi < \epsilon$.

Then, if

$$h = g' + v_n g, \quad \phi_{nn} h = \frac{d}{dx} (\phi_{nn} g),$$

h is of $L^2(0,1)$; also

$$\int_0^1 h \phi_{nn} d\xi = [g \phi_{nn}]_0^1 = 0,$$

so that, assuming the completeness of the family ϕ_{ns} , we have

$$h = \sum_{s=n+1}^N c_s \phi_{ns} + \eta,$$

where

$$\int_0^1 |\eta|^2 dx < \epsilon.$$

Now

$$\phi_{nn} g = \int_0^x \phi_{nn} h d\xi = \sum_{s=n+1}^N c_s \int_0^x \phi_{nn} \phi_{ns} d\xi + \int_0^x \phi_{nn} \eta d\xi = \phi_{nn} \sum_{s=n+1}^N C_s \phi_{n+1,s} + \phi_{nn} \zeta,$$

where $C_s = c_s(\lambda_n - \lambda_s)^{-1}$, and

$$\zeta = \frac{1}{\phi_{nn}} \int_0^x \phi_{nn} \eta d\xi = -\frac{1}{\phi_{nn}} \int_x^1 \phi_{nn} \eta d\xi;$$

since, by (G_n) and its analogue for $x \rightarrow 1$,

$$\int_0^x \phi_{nn}^2 dx = O(\phi_{nn}^2), \quad \int_x^1 \phi_{nn}^2 = O(\phi_{nn}^2)$$

when $x \rightarrow 0, 1$, respectively, we have by Schwartz's inequality

$$|\zeta|^2 < M_n \int_0^1 |\eta|^2 dx < M_n \epsilon, \quad \int_0^1 |\zeta|^2 dx < M_n \epsilon.$$

Hence the result.

5. Examples

(1) If $q(x) = 0$, $h^{(0)} = 0 = h^{(1)}$, then $\lambda_s = (2\pi s)^2$, $\phi_s = \cos 2\pi s x$ ($s = 0, 1, 2, \dots$). Since $v = 0$, $q_1 = q$ and

$$\phi_{1s} = \phi'_s = 2\pi s \sin 2\pi s x \quad (s = 1, 2, \dots) .$$

For $n > 1$, ϕ_{ns} is obtainable as in Example 3.

(2) If $q(x) = x^2$ and the interval is $(-\infty, \infty)$, (A) is $y'' + (\lambda - x^2)y = 0$, with $\phi_0 = e^{-\frac{1}{2}x^2}$, $\lambda_0 = 1$. Since $v = x$, $q_1 = x^2 - 2$; hence [2]

$$\lambda_{s+1} = \lambda_s + 2 , \quad \phi_{1s} = k_s \phi_{s-1} .$$

The associated systems are all identical, $\lambda_s = 2s + 1$, and, since

$$\phi_0 \phi_s = \frac{1}{\lambda_0 - \lambda_s} \frac{d}{dx} (\phi_0 \phi_{1s}) = \frac{k_s}{2s} \frac{d}{dx} (\phi_0 \phi_{s-1}) ,$$

it follows that

$$\phi_s = K_s \phi_0^{-1} \left(\frac{d}{dx} \right)^s \phi_0^2 = K_s e^{\frac{1}{2}x^2} \left(\frac{d}{dx} \right)^s e^{-x^2} .$$

(3) The Legendre functions [3]

$$y_s = (\sin \theta)^{\frac{1}{2}} P_s(\cos \theta) \quad (0 < \theta < \pi)$$

satisfy

$$y'' + \left(\lambda + \frac{1}{4} \operatorname{cosec}^2 \theta \right) y = 0 ,$$

where

$$\lambda_s = \left(s + \frac{1}{2} \right)^2 \quad (s = 0, 1, 2, \dots) .$$

Writing $\mu = \cos \theta$, and $W_{(\mu)}$ for the Wronskians with respect to μ , we get

$$\begin{aligned} W_n &= W(y_0, y_1, \dots, y_{n-1}) = \left(\frac{d\mu}{d\theta} \right)^{\frac{1}{2}n(n-1)} W_{(\mu)}(y_0, y_1, \dots, y_{n-1}) \\ &= \left(\frac{d\mu}{d\theta} \right)^{\frac{1}{2}n(n-1)} (\sin \theta)^{\frac{1}{2}n} W_{(\mu)}(P_0, P_1, \dots, P_{n-1}) = A_n (\sin \theta)^{\frac{1}{2}n^2} , \end{aligned}$$

and similarly

$$W_{ns} = A_n (\sin \theta)^{\frac{1}{2}(n+1)^2} \left(\frac{d}{dx} \right)^n P_s(\mu) .$$

Hence [4]

$$\phi_{ns} = (\sin \theta)^{n+\frac{1}{2}} \left(\frac{d}{dx} \right)^n P_s(\mu) = (\sin \theta)^{\frac{1}{2}} P_s^{(n)}(\mu) .$$

(4) For the Hankel system [5] of order ν

$$y = \phi_k(x) = c_k(kx)^{\frac{1}{2}} J_\nu(kx) , \quad \phi_0(x) = x^{\nu+\frac{1}{2}} ,$$

$$y'' + \left(\lambda - \frac{\nu^2 - \frac{1}{4}}{x^2} \right) y = 0 , \quad \lambda = k^2 .$$

Here $v = \phi_0' / \phi_0 = (\nu + \frac{1}{2})/x$, whence

$$q_1 = \frac{(\nu + 1)^2 - \frac{1}{4}}{x^2}$$

and the first associated system is the Hankel system of order $\nu + 1$.

6. As a corollary of the main theorem, if

$$S(x) = \sum_0^n c_s \phi_s(x) ,$$

then $S(x)$ has at most n zeros in $(0,1)$. This result is due to Kellogg [6]. For, if $S(x)$ has k zeros, then by Rolle's theorem

$$S_1(x) = \phi_0 \frac{d}{dx} \left(\phi_0^{-1} \sum_0^n c_s \phi_s(x) \right) = \sum_1^n c_s \phi_{1s}$$

has at least $k - 1$ zeros inside $(0,1)$; by induction

$$S_m(x) = \sum_m^n c_s \phi_{ms}$$

has at least $k - m$ zeros, and $S_n(x) = c_n \phi_{nn}$ has at least $k - n$; since ϕ_{nn} is non-zero, either $k \leq n$ or $c_n = 0$; but, if $c_n = 0$, then $k \leq n - 1 \leq n$.

This proof of the corollary depends only on the fact that the Wronskians W_n are non-zero. If $\phi_s = e^{\alpha_s x}$, where the α_s are any distinct real numbers, then the W_n are all non-zero, and so $S(x)$ has at most n real zeros.

7. If (A, B) is given, the associated systems (A_n, B_n) are uniquely defined; but to a given (A_n, B_n) belong an infinity of (A, B) . For example, given (A_1, B_1) we may solve for v

$$\lambda_0 - v' + v^2 = q_1 ,$$

with any λ_0 such that $\lambda_0 < \lambda_1$; then, if

$$\phi_0 = \exp \left(\int_0^x v d\xi \right) , \quad (\lambda_0 - \lambda_s) \phi_0 \phi_s = \frac{d}{dx} (\phi_0 \phi_{1s}) ,$$

it will follow that the ϕ_s are the eigenfunctions of (A, B) with

$$q = q_1 + 2v' , \quad h^{(0)} = v(0) , \quad h^{(1)} = v(1) .$$

For example, if

$$q_1 = 0, \quad \lambda_s = (2\pi s)^2, \quad \phi_{1s} = \sin 2\pi s x,$$

we can take

$$\lambda_0 = -\rho^2, \quad \phi_0 = \operatorname{sech} \rho(x - \alpha), \quad v = -\rho \tanh \rho(x - \alpha),$$

$$q(x) = -2\rho^2 \operatorname{sech}^2 \rho(x - \alpha),$$

$$\phi_s(x) = 2\pi s \cos 2\pi s x - \rho \tanh \rho(x - \alpha) \sin 2\pi s x.$$

Starting from a given (A_n, B_n) we can similarly construct an (A, B) with arbitrary $\lambda_0, \lambda_1, \dots, \lambda_{n-1}$ (provided only that $\lambda_{s+1} > \lambda_s$). Thus there exists a regular Sturm-Liouville system with any finite set of real numbers as eigenvalues.

References

1. E.L. Ince, *Ordinary Differential Equations* (London, 1927), §10.61, 235.
2. Compare P.A.M. Dirac, *Quantum Mechanics* (3rd ed., Oxford, 1947), §34, 136-139.
3. E.C. Titchmarsh, *Eigenfunction Expansions* (Oxford, 1946), §4.5, 64.
4. E.T. Whittaker and G.N. Watson, *Modern Analysis* (3rd ed., Cambridge, 1927), §15.5, 323.
5. Titchmarsh, op. cit. §4.8, 70, and §4.11, 75.
6. O.D. Kellogg, Am. J. Math. (i) *Oscillations of functions of an orthogonal set* (1916) 1, (ii) *Orthogonal sets arising from integral equations* (1918) 145, (iii) *Interpolation properties of orthogonal sets of solutions of differential equations* (1918) 225. Kellogg uses the functional determinants $\det[\phi_s(x_t)]$, not the Wronskians W_{ns} or W_{ns}^* .